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Existence and Welfare Properties of Equilibrium in an Exchange Economy with Multiple Divisible, Indivisible Commodities and Linear Production Technologies ¹

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Abstract

In this paper we consider a class of economies with a finite number of divisible commodities, linear production technologies, and indivisible goods, and a finite number of agents. This class contains several well-known economies with indivisible goods and money as special cases. It is shown that if the utility functions are continuous on the divisible commodities and are weakly monotonic both on one of the divisible commodities and on all the indivisible commodities, if each agent initially owns a sufficient amount of one of the divisible commodities, and if a “no-production-without-input”-like assumption on production sector holds, then there exists a competitive equilibrium for any economy in this class. The usual convexity assumption is not needed here. Furthermore, by imposing strong monotonicity on one of the divisible commodities we show that any competitive equilibrium is in the core of the economy and therefore the first theorem of welfare also holds. We further obtain a second welfare theorem stating that under some conditions a Pareto efficient allocation can be sustained by a competitive equilibrium allocation for some well-chosen redistribution of the total initial endowments.

Key words: indivisible commodities, divisible commodities, linear production, competitive equilibrium, equilibrium theorem, welfare theorem

JEL-code: D2, D4, D5, D6.

1 Introduction

Since Industrial Revolution, indivisible commodities have constituted a prominently important part of commercial commodities in most of the markets. Typical indivisible commodities are, to name a few, houses, cars, employees, airplanes, ships, trains, computers, machinery, and arts. Those goods are generally durable and expensive. Nowadays, even many divisible commodities are sold in indivisible quantities such as oil being sold in barrel as its smallest unit. Obviously, modelling economies in indivisibilities is more meaningful and realistic. However, due to the extreme nonconvexity, studying such economies stands in general a daunting challenge; see for example Koopmans and Beckman [8], Debreu [3], and Scarf [13, 14, 15]. In spite of the difficulties, we have seen a reviving interest in studying economies with indivisibilities in recent years. The models in Bikhchandani and Mamer [2], van der Laan, Talman and Yang [9], Ma [10], Bevia, Quinzii and Silva [1], and Yang [18] in one way or another generalize those in Shapley and Scarf [12], Kelso and Crawford [7], Quinzii [11], Gale [4], Kaneko and Yamamoto [6], and Yamamoto [16] from economies with one indivisible commodity and money to economies with multiple indivisible commodities and money.

In this paper we consider an exchange economy with a finite number of divisible commodities, linear production technologies, and indivisible goods, and a finite number of agents. In contrast, in the existing models above money was assumed to be the only divisible good and no production was involved. In our model, it is assumed that each agent initially owns one indivisible object and a certain amount of one of the divisible commodities, say, commodity zero, and that each agent can demand any amount of each of the divisible commodities but demands at most one indivisible object. Commodity zero is served as labour or capital and is used as input to produce the other divisible commodities. It is shown that if the utility functions are weakly monotonic on both commodity zero and the indivisible commodities, and are continuous on the divisible commodities, and if each agent initially owns in some sense a “sufficient” amount of commodity zero, then there exists a competitive equilibrium in the economy. The usual convexity assumption is not required here. Furthermore, by imposing strong monotonicity of the utility on commodity zero we show that any competitive equilibrium is in the core of the economy and therefore the first theorem of welfare holds. We also obtain the second welfare theorem stating that any Pareto efficient allocation satisfying some condition can be sustained by a competitive equilibrium allocation for some well-chosen redistribution of the total initial endowments.

The plan of the paper is as follows. In Section 2 the economic model and its conditions for the existence of equilibrium are introduced. In Section 3 we prove the existence of a competitive equilibrium in the model. The welfare properties are contained in Section 4.

2 The Model

Let $I_k = \{1, \dots, k\}$ be the set of the first k positive integers. For $k \in \mathbb{N}$, \mathbb{R}^k denotes the k -dimensional Euclidean space and \mathbb{R}_+^k its nonnegative orthant. For $i \in I_k$, $e(i)$ denotes the i -th k -dimensional unit vector. The vectors 0^k and 1^k denote the k -dimensional vectors of zeros and ones, respectively, and $E^k = \{0^k, e(1), \dots, e(k)\}$ is the set of all k -dimensional unit vectors and the k -dimensional vector of zeros.

We consider an exchange economy with m agents, $n + 1$ divisible commodities, m indivisible objects, and a production sector. The divisible commodities are indexed by $j = 0, 1, \dots, n$. Each agent i , $i \in I_m$, initially owns one indivisible object denoted by object i , and a certain positive amount ω_0^i of commodity zero. In the following, for agent $i \in I_m$, his bundle of initial endowments is denoted by the pair $(\omega^i, e(i))$ where $\omega^i \in \mathbb{R}_+^{n+1}$ with $\omega_j^i = 0$ for $j = 1, \dots, n$, denotes his initial endowment of the divisibilities and the unit vector $e(i)$ denotes his initial endowment of the indivisibilities, meaning that agent i only owns object i . Commodity zero can be interpreted as labour or capital. A consumption bundle of agent i is given by the pair $(x^i, \ell^i) \in \mathbb{R}_+^{n+1} \times E^m$, where the vector x^i denotes his consumption of the divisibilities and the vector ℓ^i his consumption of the indivisible objects. Note that either ℓ^i is equal to 0^m in which case agent i does not consume any object, or for some $j \in I_m$ it holds that $\ell^i = e(j)$, indicating that agent i consumes the object initially owned by agent j . The preferences of agent $i \in I_m$ are represented by a utility function $u^i: \mathbb{R}_+^{n+1} \times E^m \rightarrow \mathbb{R}$. This means that the agent derives utility from at most one indivisible object.

The production sector of the economy is specified by n linear production technologies. Production technology j , $j \in I_n$, is represented by a vector $a^j \in \mathbb{R}^{n+1}$. Let $A = [a^1, \dots, a^n]$ denote the $(n + 1) \times n$ matrix of input-output vectors of the divisible goods, let the row vector A_0 denote the first row of A , and let \hat{A} denote the $n \times n$ matrix obtained by deleting the first row from A . Production only concerns the divisibilities. The production levels are denoted by $y \in \mathbb{R}_+^n$, i.e. $y_j \geq 0$ is the level of production of production activity $j \in I_n$. For given vector y of production levels, the vector of total inputs and outputs is given by Ay .

We denote an economy as specified above by $\mathcal{E} = \{(\omega^i, e(i), u^i)_{i \in I_m}, A\}$. The prices of the divisible commodities are denoted by the vector $p = (p_0, p_1, \dots, p_n)^\top \in \mathbb{R}_+^{n+1} \setminus \{0^{n+1}\}$ and the prices of the indivisibilities by the vector $q = (q_1, \dots, q_m)^\top \in \mathbb{R}_+^m$. For the pair (p, q) of price vectors, the budget set $B^i(p, q)$ of agent i is the set of consumption bundles given by

$$B^i(p, q) = \{(x^i, \ell^i) \in \mathbb{R}_+^{n+1} \times E^m \mid p^\top x^i + q^\top \ell^i \leq p^\top \omega^i + q^\top e(i) = p_0 \omega_0^i + q^\top e(i)\}.$$

All agents are assumed to be utility maximizing agents. So given (p, q) , agent i maximizes

utility over his budget set, yielding the demand set $D^i(p, q)$ given by

$$D^i(p, q) = \{(x^i, \ell^i) \in B^i(p, q) \mid u^i(x^i, \ell^i) = \max_{(x, \ell) \in B^i(p, q)} u^i(x, \ell)\}.$$

Definition 2.1

A **competitive equilibrium** in the economy $\mathcal{E} = \{(\omega^i, e(i), u^i)_{i \in I_m}, A\}$ is a pair (p^*, q^*) of price vectors, a vector y^* of production levels, and a collection (x^{*i}, ℓ^{*i}) , $i \in I_m$, of consumption bundles, satisfying

- (i) $(x^{*i}, \ell^{*i}) \in D^i(p^*, q^*)$ for all $i \in I_m$;
- (ii) $p^{*\top} A \leq 0^n$, $y^* \geq 0^n$ and $p^{*\top} A y^* = 0$;
- (iii) $\sum_{i \in I_m} \ell^{*i} = 1^m$;
- (iv) $\sum_{i \in I_m} x^{*i} = A y^* + \sum_{i \in I_m} \omega^i$.

The first condition states that in equilibrium each agents maximizes his utility given his budget constraint at the equilibrium prices. Condition (ii) states that no production technology can make a positive profit, all activity levels are not negative and that for any technology the profit is zero if the activity level is positive. The last two conditions clear the markets of the indivisible and divisible commodities, respectively. Observe that in equilibrium the vector $A y^*$ of inputs and outputs equals the difference between the agents' total demand and supply for the divisible commodities.

With respect to the economy \mathcal{E} the following assumptions are made.

- A1.** The activity matrix A satisfies that A_0 is a strictly negative row vector, \hat{A} is a regular $n \times n$ matrix and \hat{A}^{-1} is nonnegative.
- A2.** For every agent $i \in I_m$ the utility function u^i is weakly monotonic in commodity zero and in the indivisible goods.
- A3.** For every agent $i \in I_m$ the utility function $u^i(\cdot, \ell)$ is continuous for each given $\ell \in E^m$.
- A4.** For every agent $i \in I_m$, there exist some $x^i \in \mathbb{R}_+^{n+1}$ and some $y \in \mathbb{R}_+^n$ such that $x^i = A y + \omega^i$ and

$$u^i(x^i, e(i)) > \max_{j \in I_m} u^j(0^{n+1}, e(j)).$$

The first statement in Assumption A1 implies that capital or labour is needed as input for any production technology. So, there can be no production without input, that is $A y \geq 0$ and $y \geq 0$ imply $y = 0$. Note that any Leontief input-output matrix

satisfies the second statement of Assumption A1. This part of the assumption implies that any demand for the commodities $1, \dots, n$, denoted by a nonnegative (nonzero) vector $x_{-0} = (x_1, \dots, x_n)^\top$, can be produced, i.e. there exists a nonnegative vector $y \in \mathbb{R}_+^n$ of activities levels such that $\hat{A}y = x_{-0}$, namely $y = \hat{A}^{-1}x_{-0}$. Observe that this allows that a divisible commodity other than commodity zero is an output in some activities and serves as an input in others and therefore can be both an intermediate commodity in the production sector and a final commodity in the consumption sector. Assumptions A2 and A3 are weaker than the standard conditions in the Arrow-Debreu framework. Weak monotonicity of the utility function is only required for commodity zero and the indivisible objects. The latter means that having any object is weakly preferred to having no object. Assumption A4 says that every agent's initial endowment of commodity zero is enough to produce a vector x of divisible commodities such that the pair $(x, e(i))$ is strictly preferred to any indivisible object without divisible commodities. Together with Assumption A1 this implicitly assumes that $\omega_0^i > 0$ for all i . Of course, Assumption A4 holds if the stronger condition $\max_{j \in I_m} u^i(0^{n+1}, e(j)) < u^i(\omega^i, e(i))$ holds, i.e. if the initial endowment itself is strictly preferred to any indivisible object without divisible goods. Finally, notice that we do not make any convexity assumption on the preferences, and that some of the divisible commodities $j = 1, \dots, n$, could be divisible bads.

3 Existence of Equilibrium

In this section we prove the existence of a competitive equilibrium the economy. To do so, we first prove two lemmas.

Lemma 3.1

Let the activity matrix A satisfy A1. Then there is a unique strictly positive price vector p^ , such that $p_0^* = 1$ and $p^{*\top} A = 0^n$.*

Proof.

Consider the system of n equations $p^\top A = 0^n$. Then for $j \in I_n$, equation j yields

$$\sum_{k=0}^n p_k a_k^j = 0,$$

which can be rewritten as

$$\sum_{k=1}^n p_k a_k^j = -p_0 a_0^j.$$

Take $p_0 = 1$. Then the system becomes

$$p_{-0}^\top \hat{A} = -A_0^\top,$$

where $p_{-0} = (p_1, \dots, p_n)^\top$. From the first part of A1 we know that the n vector $-A_0^\top \in \mathbb{R}_{++}^n$ and from the second part that \hat{A}^{-1} exists and is nonnegative. Hence the latter system of equations has a unique strictly positive solution equal to $-\hat{A}^{-1}A_0^\top$. Q.E.D.

In the remaining of this paper p^* stands for the unique strictly positive price vector satisfying the conditions of Lemma 3.1, so $p^{*\top}A = 0^n$ and $p_0^* = 1$. In the next lemma we show that at p^* each agent's demand correspondence is upper semi-continuous in the prices of the indivisibilities.

Lemma 3.2

Let the economy $\mathcal{E} = \{(\omega^i, e(i), u^i)_{i \in I_m}, A\}$ satisfy Assumptions A1, A3 and A4. Then, for every agent $i \in I_m$, the correspondence $D^i(p^, \cdot)$ is upper semi-continuous (u.s.c) in $q \in \mathbb{R}_+^m$.*

Proof.

Let q be an arbitrary price vector in \mathbb{R}_+^m and $i \in I_m$ an arbitrarily taken agent. Take any sequence $\{q^k\}_{k \in \mathbb{N}}$ in \mathbb{R}_+^m converging to q and sequence $\{(x^k, \ell^k)\}_{k \in \mathbb{N}}$ converging to (x, ℓ) with $(x^k, \ell^k) \in D^i(p^*, q^k)$ for all $k \in \mathbb{N}$. We have to show that $(x, \ell) \in D^i(p^*, q)$. Without loss of generality we can assume that $\ell^k = \ell$ for all $k \in \mathbb{N}$, for E^m is a finite set. Since, for all $k \in \mathbb{N}$,

$$p^{*\top}x^k + q^{k\top}\ell \leq p^{*\top}\omega^i + q^{k\top}e(i),$$

we have that

$$p^{*\top}x + q^\top\ell \leq p^{*\top}\omega^i + q^\top e(i),$$

i.e. (x, ℓ) lies in $B^i(p^*, q)$. Now, suppose $(x, \ell) \notin D^i(p^*, q)$. Then there exists $(\bar{x}, \bar{\ell}) \in D^i(p^*, q)$ such that

$$u^i(\bar{x}, \bar{\ell}) > u^i(x, \ell) \tag{1}$$

and

$$p^{*\top}\bar{x} + q^\top\bar{\ell} \leq p^{*\top}\omega^i + q^\top e(i). \tag{2}$$

We will prove that $\bar{x} \neq 0^{n+1}$. Suppose that $\bar{x} = 0^{n+1}$. Then from Assumption A4 it follows that there exists a vector \tilde{x} such that

$$u^i(\bar{x}, \bar{\ell}) < u^i(\tilde{x}, e(i))$$

and

$$\tilde{x} = A\tilde{y} + \omega^i$$

for some $\tilde{y} \in \mathbb{R}_+^n$. From the last equation we obtain

$$p^{*\top} \tilde{x} = p^{*\top} A \tilde{y} + p^{*\top} \omega^i = p^{*\top} \omega^i,$$

and so $(\tilde{x}, e(i)) \in B^i(p^*, q)$. This contradicts $(\bar{x}, \bar{\ell}) \in D^i(p^*, q)$ and therefore we must have that $\bar{x} \neq 0^{n+1}$. With Assumption A3 and from equations (1) and (2) it now follows that there exists $\varepsilon > 0$ such that $\tilde{x} = (1 - \varepsilon)\bar{x} \in \mathbb{R}_+^{n+1}$ satisfies

$$u^i(\tilde{x}, \bar{\ell}) > u^i(x, \ell) \quad (3)$$

and

$$p^{*\top} \tilde{x} + q^\top \bar{\ell} < p^{*\top} \omega^i + q^\top e(i).$$

From the latter inequality it follows that for k large enough it holds that

$$p^{*\top} \tilde{x} + q^{k\top} \bar{\ell} < p^{*\top} \omega^i + q^{k\top} e(i) \quad (4)$$

and from A3 and inequality (3) it follows that for k large enough

$$u^i(\tilde{x}, \bar{\ell}) > u^i(x^k, \ell). \quad (5)$$

Together the inequalities (4) and (5) contradict $(x^k, l) \in D^i(p^*, q^k)$ for k large enough. Hence we must have that $(x, l) \in D^i(p^*, q)$ and so $D^i(p^*, \cdot)$ is upper semi-continuous.

Q.E.D.

We are now ready to state the main equilibrium existence theorem.

Theorem 3.3

Let the economy $\mathcal{E} = \{(\omega^i, e(i), u^i)_{i \in I_m}, A\}$ satisfy Assumptions A1-A4. Then there exists a competitive equilibrium.

Proof. First, for some real number $M > \sum_{i \in I_m} \omega_0^i$, define the m -dimensional set U^M by

$$U^M = \{q \in \mathbb{R}_+^m \mid q_j \leq M \text{ for all } j \in I_m\}.$$

From Lemma 3.2 we have that for each agent i the demand correspondence $D^i(p^*, \cdot)$ is u.s.c on U^M . For $i \in I_m$, define $D_{ind}^i(p^*, q)$ as the demand set restricted to the indivisibilities, i.e.

$$D_{ind}^i(p^*, q) = \{\ell \in E^m \mid (x, \ell) \in D^i(p^*, q) \text{ for some } x \in \mathbb{R}_+^{n+1}\}.$$

Further, define $\widetilde{D}_{ind}(p^*, q)$ as the sum of the convex hulls of these sets, i.e.

$$\widetilde{D}_{ind}(p^*, q) = \sum_{i \in I_m} \text{Conv}(D_{ind}^i(p^*, q)),$$

where $\text{Conv}(\cdot)$ denotes the convex hull. Next, let $Z(p^*, q)$ be defined by

$$Z(p^*, q) = \widetilde{D}_{ind}(p^*, q) - \{1^m\}.$$

Clearly, the correspondence $Z(p^*, \cdot)$ is convex, compact and non-empty valued and upper semi-continuous at any $q \in U^M$. From Yang [17] it follows that there exist an m -vector $q^* \in U^M$ and an m -vector $\tilde{\ell} \in \widetilde{D}_{ind}(p^*, q^*)$ such that $\tilde{\ell}$ satisfies the conditions

$$\begin{aligned} \ell_j - 1 &\leq 0 & \text{if } q_j^* = 0, \\ \ell_j - 1 &= 0 & \text{if } 0 < q_j^* < M, \\ \ell_j - 1 &\geq 0 & \text{if } q_j^* = M. \end{aligned}$$

Consequently, there exists a tuple $\{\tilde{\ell}^1, \dots, \tilde{\ell}^m\}$ of m -dimensional vectors with $\tilde{\ell}^i \in \text{Conv}(D_{ind}^i(p^*, q^*))$ for all $i \in I_m$ and $\sum_{i \in I_m} \tilde{\ell}^i = \tilde{\ell}$, that satisfies the conditions

$$\begin{aligned} \sum_{i=1}^m \ell_j^i &\leq 1 & \text{if } q_j^* = 0, \\ \sum_{i=1}^m \ell_j^i &= 1 & \text{if } 0 < q_j^* < M, \\ \sum_{i=1}^m \ell_j^i &\geq 1 & \text{if } q_j^* = M. \end{aligned} \tag{6}$$

For $i, j \in I_m$, let c^i be the vector in \mathbb{R}_+^m defined by

$$\begin{aligned} c_j^i &= 1 & \text{if } e(j) \in D_{ind}^i(p^*, q^*), \\ c_j^i &= 0 & \text{otherwise.} \end{aligned} \tag{7}$$

Since for all $i \in I_m$ it holds that $\tilde{\ell}^i \in \text{Conv}(D_{ind}^i(p^*, q^*))$, it follows that the collection $\{\tilde{\ell}^1, \dots, \tilde{\ell}^m\}$ also satisfies the conditions

$$\begin{aligned} \sum_{j=1}^m \ell_j^i &= 1 & \text{if } 0^m \notin D_{ind}^i(p^*, q^*), \\ \sum_{j=1}^m \ell_j^i &\leq 1 & \text{otherwise,} \\ 0 \leq \ell_j^i &\leq c_j^i & \text{for all } i, j \in I_m. \end{aligned} \tag{8}$$

From Hoffman and Kruskal [5] we know that the set of m^2 variables ℓ_j^i satisfying the (in)equalities given in (6) and (8) has an integral solution. Let the collection $\{\hat{\ell}^1, \dots, \hat{\ell}^m\}$ denote such an integral solution. From the conditions given in (8) it follows that for every i the vector $\hat{\ell}^i$ contains at most one component equal to one. Suppose that $\hat{\ell}_j^i = 1$ for some $j \in I_m$. Then it follows from the first and third restrictions in (8) and from the definition of c_j^i in (7) that $e(j) \in D_{ind}^i(p^*, q^*)$ and hence $\hat{\ell}^i = e(j) \in D_{ind}^i(p^*, q^*)$. Now, suppose that $\hat{\ell}^i = 0^m$. Then it follows from the first two restrictions in (8) that $0^m \in D_{ind}^i(p^*, q^*)$ and hence $\hat{\ell}^i \in D_{ind}^i(p^*, q^*)$. So, for all $i \in I_m$ we have that $\hat{\ell}^i \in D_{ind}^i(p^*, q^*)$.

We next show that the collection $\{\hat{\ell}^1, \dots, \hat{\ell}^m\}$ satisfies $\sum_{i \in I_m} \hat{\ell}^i \leq 1^m$. Define

$$\begin{aligned} J^1 &= \{j \in I_m \mid \sum_{i \in I_m} \hat{\ell}_j^i = 0\}, \\ J^2 &= \{j \in I_m \mid \sum_{i \in I_m} \hat{\ell}_j^i = 1\}, \\ J^3 &= \{j \in I_m \mid \sum_{i \in I_m} \hat{\ell}_j^i > 1\}. \end{aligned}$$

We will show that $J^3 = \emptyset$. For $i \in I_m$, let \hat{x}^i be a vector of divisible goods such that $(\hat{x}, \hat{\ell}^i) \in D^i(p^*, q^*)$. Recall that $\hat{\ell}^i \in D_{ind}^i(p^*, q^*)$ and hence such a vector \hat{x}^i exists. From the budget constraint it follows that

$$p^{*\top} \hat{x}^i + q^{*\top} \hat{\ell}^i \leq p^{*\top} \omega^i + q^{*\top} e(i) = \omega_0^i + q^{*\top} e(i), \quad i \in I_m.$$

Adding up over all $i \in I_m$ we obtain

$$\sum_{i=1}^m p^{*\top} \hat{x}^i + \sum_{i=1}^m q^{*\top} \hat{\ell}^i \leq \sum_{i=1}^m \omega_0^i + \sum_{i=1}^m q^{*\top} e(i).$$

Therefore,

$$\sum_{i=1}^m p^{*\top} \hat{x}^i + \sum_{i=1}^m q^{*\top} (\hat{\ell}^i - e(i)) \leq \sum_{i=1}^m \omega_0^i.$$

Because $q_j^* = 0$ if $j \in J^1$ and $\sum_{i=1}^m \hat{\ell}_j^i = 1 = \sum_{i=1}^m e_j(i)$ if $j \in J^2$ it follows that

$$\sum_{j \in J^3} q_j^{*\top} \left(\sum_{i=1}^m \hat{\ell}_j^i - 1 \right) \leq \sum_{i=1}^m \omega_0^i - \sum_{i=1}^m p^{*\top} \hat{x}^i.$$

Consequently,

$$\sum_{j \in J^3} q_j^{*\top} \leq \sum_{j \in J^3} q_j^{*\top} \left(\sum_{i=1}^m \hat{\ell}_j^i - 1 \right) \leq \sum_{i=1}^m \omega_0^i < M.$$

Together with the third restriction of system (6) it follows that $J^3 = \emptyset$ and hence $\sum_{i=1}^m \hat{\ell}_j^i \leq 1$ for all $j \in I_m$.

We now consider the indivisible goods for which $j \in J^1$, i.e. $\sum_{i=1}^m \hat{\ell}_j^i = 0$. First notice that the collection $\{\hat{\ell}^1, \dots, \hat{\ell}^m\}$ assigns the indivisible goods $j \in J^2$ to $|J^2|$ different agents, leaving the other $m - |J^2| = |J^1|$ agents without any object. Furthermore it follows from the first condition in (6) that $q_j^* = 0$ if $j \in J^1$. Since by Assumption A2 the utility functions are weakly monotonic in the indivisible goods it therefore must hold that

$$u^i(\hat{x}^i, e(j)) = u^i(\hat{x}^i, 0^m) \text{ if } j \in J^1 \text{ and } \hat{\ell}^i = 0^m,$$

and therefore

$$(\hat{x}^i, e(j)) \in D^i(p^*, q^*) \text{ if } j \in J^1 \text{ and } \hat{\ell}^i = 0^m.$$

From this it follows that the indivisible goods in J^1 can be assigned to the agents i for which $\hat{\ell}^i = 0^m$ by any arbitrarily chosen one to one assignment. Let $i(j)$ be the agent in the set $I^0 = \{i \in I_m \mid \hat{\ell}^i = 0^m\}$ who has been assigned object $j \in J^1$ in this way. Then we define the collection $\{\ell^{*1}, \dots, \ell^{*m}\}$ by

$$\begin{aligned} \ell^{*i} &= \hat{\ell}^i && \text{if } i \notin I^0, \\ \ell^{i(j)} &= e(j) && \text{if } i(j) \in I^0. \end{aligned}$$

It follows that $(\hat{x}^i, \ell^{*i}) \in D^i(p^*, q^*)$ for all $i \in I_m$. Furthermore

$$\sum_{i=1}^m \ell^{*i} = 1^m$$

and so all markets of the indivisible goods clear.

We finally define a collection $\{x^{*1}, \dots, x^{*m}\}$ of $(n+1)$ -vectors of divisible goods. When for $i \in I_m$ the budget restriction

$$p^{*\top} \hat{x}^i + q^{*\top} \ell^{*i} \leq p^{*\top} \omega^i + q^{*\top} e(i)$$

holds with equality we define $x^{*i} = \hat{x}^i$. When the budget restriction holds with inequality we define $x_j^{*i} = \hat{x}_j^i$ for all $j \in I_n$ and $x_0^{*i} > \hat{x}_0^i$ such that

$$p^{*\top} x^{*i} + q^{*\top} \ell^{*i} = p^{*\top} \omega^i + q^{*\top} e(i). \quad (9)$$

From the weak monotonicity assumption with respect to good 0 (Assumption A2) it follows that $(x^{*i}, \ell^{*i}) \in D^i(p^*, q^*)$ for all $i \in I_m$. Since equation (9) holds for all $i \in I_m$, we obtain that

$$p^{*\top} \sum_{i=1}^m x^{*i} + q^{*\top} \sum_{i=1}^m \ell^{*i} = p^{*\top} \sum_{i=1}^m \omega^i + q^{*\top} 1^m.$$

Hence,

$$p^{*\top} \sum_{i=1}^m x^{*i} = p^{*\top} \sum_{i=1}^m \omega^i \quad (10)$$

since we already showed that the markets of the indivisibilities clear. We now define the vector y^* of activity levels by

$$y^* = \hat{A}^{-1} x_{-0}^*, \quad (11)$$

where $x_{-0}^* = (\sum_{i=1}^m x_1^{*i}, \dots, \sum_{i=1}^m x_n^{*i})^\top$. Since $x_{-0}^* \in \mathbb{R}_+^n$ it follows from Assumption A1 that $y^* \in \mathbb{R}_+^n$. Hence y^* is a feasible vector of the production activity levels. From equation (10) and $p^{*\top} A = 0^n$ it follows that

$$p^{*\top} \sum_{i=1}^m x^{*i} = p^{*\top} A y^* + p^{*\top} \sum_{i=1}^m \omega^i. \quad (12)$$

Since equation (11) implies that $x_{-0}^* = \hat{A} y^*$ and hence all markets $j \in I_n$ are in equilibrium, it follows from equation (12) that also the market of commodity 0 is in equilibrium. This proves that the tuple $\{p^*, q^*, y^*, (x^{*i}, \ell^{*i})_{i \in I_m}\}$ is a competitive equilibrium. Q.E.D.

If we apply the methods described in Yang [17], we can actually compute a competitive equilibrium in the economy. This is quite desirable in practice. It should be also

observed that there are probably multiple integer solutions to the systems of (in)equalities (6) and (8). Any solution results in a distribution of the indivisible commodities over the agents. Two different equilibrium distributions induced by two of such integer solutions may result in different corresponding excess demand vectors for the divisibilities. Nevertheless the conditions on the model guarantee that any total demand for the divisibilities at such an equilibrium distribution of the indivisibilities can be produced by the production sector.

4 Welfare Theorems

In this section we analyse the welfare properties of the competitive equilibria for the economy $\mathcal{E} = \{(\omega^i, \epsilon(i), u^i)_{i \in I_m}, A\}$. First we give the definitions of a feasible allocation and Pareto efficiency. Given any $x^1, \dots, x^m \in \mathbb{R}_+^{n+1}$ and $\ell^1, \dots, \ell^m \in E^m$, we call the m -tuple $(x, \ell) = ((x^1, \ell^1), \dots, (x^m, \ell^m))$ of pairs an allocation. We first give the definitions of a feasible allocation and a Pareto efficient allocation.

Definition 4.1

An allocation (x, ℓ) is feasible if

- (i) $\sum_{i=1}^m x^i = Ay + \sum_{i=1}^m \omega^i$ for some $y \in \mathbb{R}_+^n$,
- (ii) $\sum_{i=1}^m \ell^i = 1^m$.

The definition says that an allocation is feasible if it induces a redistribution of the indivisibilities and if the excess demand vector can be produced by the production sector.

Definition 4.2

An allocation (x, ℓ) is Pareto efficient if it is feasible and there does not exist a feasible allocation (x', ℓ') such that

- (i) $u^i(x'^i, \ell'^i) \geq u^i(x^i, \ell^i)$ for all $i \in I_m$,
- (ii) $u^i(x'^i, \ell'^i) > u^i(x^i, \ell^i)$ for at least one $i \in I_m$.

Because in Assumption A2 we only require weak monotonicity, in general a competitive equilibrium allocation can not be guaranteed to be Pareto efficient according to Definition 4.2. To assure this we could use the weaker requirement that (x, ℓ) is said to be Pareto efficient if there does not exist a feasible allocation (x', ℓ') such that $u^i(x'^i, \ell'^i) > u^i(x^i, \ell^i)$ for all $i \in I_m$. Instead of doing this, in this section we replace Assumption A2 by a slightly stronger assumption:

A'2. For every agent $i \in I_m$ the utility function u^i is strongly monotonic in commodity zero and weakly monotonic in the indivisible goods.

To show that the competitive equilibrium allocation is Pareto efficient we will prove that this allocation is in the core of the economy. We therefore first give a definition of the core. Then we show that the core of the economy is non-empty by proving that any competitive equilibrium allocation with the price vector p^* for the divisible commodities lies in the core and therefore is also Pareto efficient. To give a definition of the core, we first state the concept of the domination of an allocation.

Definition 4.3

A subset S of the set I_m of agents is able to dominate an allocation (x, ℓ) if there exists an allocation (x', ℓ') such that

- (i) $\sum_{i \in S} x'^i = Ay' + \sum_{i \in S} \omega^i$ for some $y' \in \mathbb{R}_+^n$,
- (ii) $\sum_{i \in S} \ell'^i = \sum_{i \in S} e(i)$,
- (iii) $u^i(x'^i, \ell'^i) \geq u^i(x^i, \ell^i)$ for all $i \in S$ and $u^i(x'^i, \ell'^i) > u^i(x^i, \ell^i)$ for some $i \in S$.

An allocation (x, ℓ) is said to be dominated if there is an $S \subset I_m$ of agents that is able to dominate (x, ℓ) .

If a subset S of agents is able to dominate the allocation (x, ℓ) then they are able to improve on (x, ℓ) , i.e. making at least one of them better off and all others at least equally well, by utilizing the production structure for production of divisibilities out of their own resources of commodity 0 and trading among themselves the indivisibilities. So, we allow any coalition to make use of the full production possibilities, i.e. the production activities are non-excludable. This is reasonable given the constant-returns-to-scale structure of the production sector.

Definition 4.4

An allocation (x, ℓ) is in the core of the economy $\mathcal{E} = \{(\omega^i, e(i), u^i)_{i \in I_m}, A\}$ if it is feasible and there does not exist a subset S of I_m that is able to dominate (x, ℓ) .

Theorem 4.5

Let the economy $\mathcal{E} = \{(\omega^i, e(i), u^i)_{i \in I_m}, A\}$ satisfy Assumptions A1 and A'2 and let (x^*, ℓ^*) be a competitive equilibrium allocation with p^* as its equilibrium prices of divisible commodities. Then (x^*, ℓ^*) is a core allocation.

Proof. Let (p^*, q^*) be the corresponding competitive price system with respect to (x^*, ℓ^*) . From Lemma 3.1, we have $p^{*\top} A = 0^n$. Suppose that (x^*, ℓ^*) is not in the core. Then there exist a subset S of I_m and an allocation (x, ℓ) such that

- (i) $\sum_{i \in S} x^i = Ay + \sum_{i \in S} \omega^i$ for some $y \in \mathbb{R}_+^n$,
- (ii) $\sum_{i \in S} \ell^i = \sum_{i \in S} e(i)$,
- (iii) $u^i(x^i, \ell^i) \geq u^i(x^{*i}, \ell^{*i})$ for all $i \in S$ and $u^i(x^i, \ell^i) > u^i(x^{*i}, \ell^{*i})$ for some $i \in S$.

From the utility maximization behavior it follows that $(x^{*i}, \ell^{*i}) \notin B^i(p^*, q^*)$ for all $i \in S$ with $u^i(x^i, \ell^i) > u^i(x^{*i}, \ell^{*i})$, i.e.

$$p^{*\top} x^i + q^{*\top} \ell^i > p^{*\top} \omega^i + q^{*\top} e(i).$$

Because of the strong monotonicity (Assumption A'2) with respect to commodity zero it must hold that for every agent $i \in S$ with $u^i(x^i, \ell^i) = u^i(x^{*i}, \ell^{*i})$,

$$p^{*\top} x^i + q^{*\top} \ell^i \geq p^{*\top} \omega^i + q^{*\top} e(i).$$

Since at least one agent in S is made better off, adding up over all $i \in S$ yields

$$p^{*\top} \sum_{i \in S} x^i + q^{*\top} \sum_{i \in S} \ell^i > p^{*\top} \sum_{i \in S} \omega^i + q^{*\top} \sum_{i \in S} e(i)$$

and hence

$$p^{*\top} \sum_{i \in S} x^i > p^{*\top} \sum_{i \in S} \omega^i$$

because of equality (ii). However, (i) implies that

$$p^{*\top} \sum_{i \in S} x^i = p^{*\top} Ay + p^{*\top} \sum_{i \in S} \omega^i = p^{*\top} \sum_{i \in S} \omega^i$$

because $p^{*\top} Ay = 0$. Hence a contradiction has been obtained. Q.E.D.

We now have the following core allocation existence theorem.

Theorem 4.6

Let the economy $\mathcal{E} = \{(\omega^i, e(i), u^i)_{i \in I_m}, A\}$ satisfy Assumptions A1, A'2, A3 and A4. Then the core is not empty.

Proof.

In the proof of Theorem 3.3 we have shown that there exists a competitive equilibrium $(p^*, q^*, y^*, (x^{*i}, \ell^{*i})_{i \in I_m})$ with p^* the unique strictly positive price vector satisfying $p^{*\top} A = 0^n$. From Theorem 4.5 it follows that the allocation (x^*, ℓ^*) lies in the core. So, the core is not empty. Q.E.D.

Clearly, an allocation in the core is Pareto efficient, because no coalition is able to dominate a core allocation and hence neither is the grand coalition of agents I_m . So, a core allocation satisfies Definition 4.2 of Pareto efficiency. This gives us immediately the **First Theorem of Welfare**, stated in the next theorem.

Theorem 4.7

Let the economy $\mathcal{E} = \{(\omega^i, e(i), u^i)_{i \in I_m}, A\}$ satisfy Assumptions A1 and A'2 and let (x^*, ℓ^*) be a competitive equilibrium allocation with p^* as its equilibrium prices of divisible commodities. Then (x^*, ℓ^*) is Pareto efficient.

Finally we prove that under some conditions also the **Second Theorem of Welfare** holds by using the existence result of Section 3. This approach offers us a simple proof of the second welfare theorem. We first prove the following lemma.

Lemma 4.8

Let $(\hat{x}, \hat{\ell})$ be a feasible allocation for the economy $\mathcal{E} = \{(\omega^i, e(i), u^i)_{i \in I_m}, A\}$. Then the allocation $(\hat{\omega}, \hat{\ell})$ defined by $\hat{\omega}^i = (p^{*\top} \hat{x}^i, 0, \dots, 0)^\top$, $i \in I_m$, is a redistribution of the initial endowments $(\omega^i, e(i))_{i \in I_m}$.

Proof.

Since $(\hat{x}, \hat{\ell})$ is a feasible allocation, it holds that

$$\sum_{i=1}^m \hat{\ell}^i = 1^m \quad (13)$$

and that there exists $\hat{y} \in \mathbb{R}_+^n$ such that

$$\sum_{i=1}^m \hat{x}^i = A\hat{y} + \sum_{i=1}^m \omega^i. \quad (14)$$

Moreover, by the definition of p^* , we have that $p^{*\top} A\hat{y} = 0$. So, premultiplying equation (14) with p^* yields

$$\sum_{i=1}^m p^{*\top} \hat{x}^i = p^{*\top} A\hat{y} + \sum_{i=1}^m p^{*\top} \omega^i = \sum_{i=1}^m \omega_0^i,$$

while also by definition of $\hat{\omega}_0^i$, $i \in I_m$ it holds that $\sum_{i=1}^m \hat{\omega}_0^i = \sum_{i=1}^m p^{*\top} \hat{x}^i$. Hence

$$\sum_{i=1}^m \hat{\omega}^i = \sum_{i=1}^m \omega^i.$$

Together with equation (13) this shows that $(\hat{\omega}^i, \hat{\ell}^i)_{i \in I_m}$ is a redistribution of the initial endowments $(\omega^i, e(i))_{i \in I_m}$. Q.E.D.

By using Lemma 4.8, we will prove the second theorem of welfare next.

Theorem 4.9

Let the economy $\mathcal{E} = \{(\omega^i, e(i), u^i)_{i \in I_m}, A\}$ satisfy Assumptions A1-A3 and let $(\hat{x}, \hat{\ell})$ be

a Pareto efficient allocation such that for every $i \in I_m$, there exist some $x^i \in \mathbb{R}_+^{n+1}$ and some $y \in \mathbb{R}_+^n$ satisfying $x^i = Ay + \hat{\omega}^i$ and

$$\max_{j \in I_m} u^i(0^{n+1}, e(j)) < u^i(x^i, \hat{\ell}^i)$$

where $\hat{\omega}^i = (p^{*\top} \hat{x}^i, 0, \dots, 0)^\top$. Then $(\hat{x}, \hat{\ell})$ is a competitive equilibrium allocation for the economy $\hat{\mathcal{E}} = \{(\hat{\omega}^i, \hat{\ell}^i, u^i)_{i \in I_m}, A\}$.

Proof. First of all, notice that $\hat{x}^i \neq 0^{n+1}$, otherwise the condition about the utilities can not hold. The condition on the utilities implies that the economy $\hat{\mathcal{E}}$ satisfies Assumptions A1-A4. From Theorem 3.3 it follows that $\hat{\mathcal{E}}$ has a competitive equilibrium $(\hat{p}^*, \hat{q}^*, \hat{y}^*, (\hat{x}^*, \hat{\ell}^*))$ satisfying

- (i) $\hat{p}^* = p^*$,
- (ii) $\sum_{i=1}^m \hat{x}^{*i} = A\hat{y}^* + \sum_{i=1}^m \hat{\omega}^i$,
- (iii) $(\hat{x}^{*i}, \hat{\ell}^{*i}) \in D^i(\hat{p}^*, \hat{q}^*)$ for all $i \in I_m$,
- (iv) $\sum_{i=1}^m \hat{\ell}^{*i} = 1^m$.

Since $p^{*\top} \hat{x}^i = \hat{\omega}_0^i = p^{*\top} \hat{\omega}^i$, we obtain for all $i \in I_m$ that

$$p^{*\top} \hat{x}^i + q^{*\top} \hat{\ell}^i = p^{*\top} \hat{\omega}^i + q^{*\top} \hat{\ell}^i,$$

and hence $(\hat{x}^i, \hat{\ell}^i) \in B^i(p^*, q^*)$. Then from (iii) it follows that

$$u^i(\hat{x}^{*i}, \hat{\ell}^{*i}) \geq u^i(\hat{x}^i, \hat{\ell}^i), \quad i \in I_m. \quad (15)$$

From Lemma 4.8 we have that the allocation $(\hat{\omega}^i, \hat{\ell}^i)_{i \in I_m}$ is a redistribution of the initial endowments $(\omega^i, e(i))_{i \in I_m}$. So, an allocation (x, ℓ) is feasible for the economy $\hat{\mathcal{E}}$ if and only if it is feasible for the economy \mathcal{E} . Clearly, this also holds for Pareto efficient allocations. So, since $(\hat{x}, \hat{\ell})$ is Pareto efficient for \mathcal{E} it is also a Pareto efficient allocation for $\hat{\mathcal{E}}$. Now, suppose there exists an $i \in I_m$ such that (15) holds with strict inequality. Then this contradicts the Pareto efficiency of $(\hat{x}, \hat{\ell})$ for $\hat{\mathcal{E}}$. So, for all $i \in I_m$ we have equality in (15) and hence $(\hat{x}^i, \hat{\ell}^i) \in D^i(p^*, q^*)$ for all $i \in I_m$. Since $(\hat{x}, \hat{\ell})$ is a feasible allocation for $\hat{\mathcal{E}}$, there exists $\hat{y} \in \mathbb{R}_+^n$ such that

$$\sum_{i=1}^m \hat{x}^i = A\hat{y} + \sum_{i=1}^m \hat{\omega}^i.$$

Since by definition of p^* , also $p^{*\top} A = 0^n$ holds, it follows that $(p^*, \hat{q}, \hat{y}, (\hat{x}, \hat{\ell}))$ satisfies all the equilibrium conditions of Definition 2.1 for the economy $\hat{\mathcal{E}}$. Q.E.D.

References

- [1] C. Bevia, M. Quinzii and J. Silva, “Buying several indivisible goods”, *Mathematical Social Science* 37 (1999) 1-23.
- [2] S. Bikhchandani and J.W. Mamer, “Competitive equilibrium in an exchange economy with indivisibilities”, *Journal of Economic Theory* 74 (1997) 385-413.
- [3] G. Debreu, *Theory of Value*, Yale University Press, New Haven, 1959.
- [4] D. Gale, “Equilibrium in a discrete economy with money”, *International Journal of Game Theory* 13 (1984) 61-64.
- [5] A.J. Hoffman and J.B. Kruskal, “Integral boundary points of convex polyhedra”, in: H.W. Kuhn and A.W. Tucker, eds., *Annals of Mathematical Studies* 38, Princeton University Press, Princeton, 1956.
- [6] M. Kaneko and Y. Yamamoto, “The existence and computation of competitive equilibria in markets with an indivisible commodities”, *Journal of Economic Theory* 38 (1986) 118-136.
- [7] A.S. Kelso and V.P. Crawford, “Job matching coalition formation and gross substitutes”, *Econometrica* 50 (1982) 1483-1504.
- [8] T. Koopmans and M.J. Beckman, “Assignment problems and the location of economic activities”, *Econometrica* 25 (1957) 53-76.
- [9] G. van der Laan, A.J.J. Talman and Z. Yang, “Existence of an equilibrium in a competitive economy with indivisibilities and money”, *Journal of Mathematical Economics* 28 (1997) 101-109.
- [10] J. Ma, “Competitive equilibria with indivisibilities”, *Journal of Economic Theory* 82 (1998) 458-468.
- [11] M. Quinzii, “Core and competitive equilibria with indivisibilities”, *International Journal of Game Theory* 13 (1984) 41-60.
- [12] L.S. Shapley and H. Scarf, “On cores and indivisibilities”, *Journal of Mathematical Economics* 1 (1974) 23-37.
- [13] H. Scarf, “Production sets with indivisibilities-part I: generalities”, *Econometrica* 49 (1981) 1-32.

- [14] H. Scarf, “Neighborhood systems for production sets with indivisibilities”, *Econometrica* 54 (1986) 507-537.
- [15] H. Scarf, “The allocation of resources in the presence of indivisibilities”, *Journal of Economic Perspectives* 4 (1994) 111-128.
- [16] Y. Yamamoto, “Competitive equilibria in a market with indivisibility”, in: A.J.J. Talman and G. van der Laan, eds., *The Computation and Modelling of Economic Equilibria*, North-Holland, 1987.
- [17] Z. Yang, *Computing Equilibria and Fixed Points*, Kluwer Academic Publishers, Boston, 1999.
- [18] Z. Yang, “Equilibria in an exchange economy with multiple indivisible commodities and money”, forthcoming in *Journal of Mathematical Economics*, 1999.